CONNECTIVITY OF COMPLEXES OF SEPARATING CURVES

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In memory of Fritz Grünewald (1949-2010)

ABSTRACT. We prove that the separated curve complex of a closed orientable surface of genus g is $(g-3)$-connected. We also obtain a connectivity property for a separated curve complex of the open surface that is obtained by removing a finite set from a closed one, where it is assumed that the removed set is endowed with a partition and that the separating curves respect that partition. These connectivity statements have implications for the algebraic topology of the moduli space of curves.

1. STATEMENTS OF THE RESULTS

Let S be a connected oriented surface of genus g with finite first Betti number $2g+n$ (i.e., a closed surface with n punctures) and make the customary assumption that S has negative Euler characteristic: if $g=0$, then $n\geq 3$ and if $g=1$, then $n\geq 1$. We recall that the *curve complex* $\mathcal{C}(S)$ of S is the simplicial complex whose vertex set consists of the isotopy classes of embedded (unoriented) circles in S which do not bound in S a disk or a cylinder. A finite set of vertices spans a simplex precisely when its elements can be represented by embedded circles that are pairwise disjoint. Thus, a closed 1-dimensional submanifold A of S with $k+1$ connected components such that every connected component of its complement has negative Euler characteristic defines a k-simplex σ_A of $\mathcal{C}(S)$ and every simplex of $\mathcal{C}(S)$ is thus obtained.

This complex has proven to be quite useful in the study of the mapping class group of S. For the purposes of studying the Torelli group of S a subcomplex $\mathcal{C}_{\mathrm{sep}}(S)$ of $\mathcal{C}(S)$ can render a similar service. It is defined as the full subcomplex of $\mathcal{C}(S)$ spanned by the separating vertices of $\mathcal{C}(S)$, where a vertex is called *separating* if a representative embedded circle separates S into two components. Our main result for the case when S is closed is:

Theorem 1.1 (A_g)**.** *If $n\leq 1$, then the simplicial complex $\mathcal{C}_{\mathrm{sep}}(S)$ is $(g-3)$-connected.*

Previous work on this topic that we are aware of concerns the case $n=0$. Farb and Ivanov announced in 2005 [1, Thm. 4] that $\mathcal{C}_{\mathrm{sep}}(S)$ is connected for

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$g \geq 3$. Putman gave in [4, Thm. 1.4] another proof of this and showed that $\mathcal{C}_{\text{sep}}(S)$ is simply connected for $g \geq 4$ (*op. cit.*, Thm. 1.11). In that paper he also mentions that Hatcher and Vogtmann have proved that $\mathcal{C}_{\text{sep}}(S)$ is $\lfloor \frac{1}{2}(g-3) \rfloor$-connected for all g (unpublished).

Remark 1.2. Presumably the connectivity bound in Theorem 1.1 is the best possible for every positive genus. In a paper with Van der Kallen [3] we showed that the quotient of $\mathcal{C}_{\text{sep}}(S)$ by the action of the Torelli group of S has the homotopy type of a bouquet of $(g-2)$-spheres.

Before we state a version for the case $n \geq 2$, we point out a consequence that pertains to the moduli space of curves. Consider the Teichmüller space $\mathcal{T}(S)$ of S on which acts the mapping class group $\Gamma(S)$, so that the orbit space may be identified with the moduli space $\mathcal{M}_g$ of curves of that genus. The *Harvey bordification* of $\mathcal{T}(S)$, here denoted by $\mathcal{T}(S)^+ \supset \mathcal{T}(S)$, is a (noncompact) manifold with boundary with corners to which the action of $\Gamma(S)$ naturally extends. This action is proper and the orbit space $\mathcal{M}_g^+ := \Gamma(S)\backslash\mathcal{T}(S)^+$ is a compactification of $\mathcal{M}_g$ that can also be obtained from the Deligne-Mumford compactification $\overline{\mathcal{M}}_g \supset \mathcal{M}_g$ as a 'real oriented blowup' of its boundary $\Delta_g := \overline{\mathcal{M}}_g - \mathcal{M}_g$. The walls of $\mathcal{T}(S)^+$ define a closed covering of the boundary $\partial\mathcal{T}(S)^+$ and any nonempty corner closure is an intersection of walls. As is well-known, the curve complex $\mathcal{C}(S)$ can be identified with the nerve of this covering of $\partial\mathcal{T}(S)^+$. Since the corner closures are contractible, Weil's nerve theorem implies that $\partial\mathcal{T}(S)^+$ has the same homotopy type as $\mathcal{C}(S)$.

Let $\Delta_{g,0} \subset \Delta_g$ denote the irreducible component of the Deligne-Mumford boundary whose generic point parameterizes irreducible curves with one singular point. We may understand $\mathcal{M}_g^c := \overline{\mathcal{M}}_g - \Delta_{g,0}$ as the moduli space of stable genus g curves with compact Jacobian and $\Delta_g^c := \Delta_g - \Delta_{g,0}$ as the locus in $\mathcal{M}_g^c$ that parameterizes the singular ones among them.

Corollary 1.3. *Let $\tilde{\mathcal{M}}_g^c \to \mathcal{M}_g^c$ be a (not necessarily finite) cover defined by a torsion free subgroup $\Gamma \subset \Gamma(S)$ (this means every Dehn twist along a separating curve in S has a positive power lying in Γ) and denote by $\tilde{\Delta}_g^c \subset \tilde{\mathcal{M}}_g^c$ the preimage of Δ_g^c. Then the pair $(\tilde{\mathcal{M}}_g^c, \tilde{\Delta}_g^c)$ is $(g-2)$-connected, and $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ for $k \leq g-2$.*

Proof. Let $\mathcal{T}(S)^+_{\text{sep}}$ be obtained from $\mathcal{T}(S)^+$ by removing the walls that correspond to the nonseparating vertices of $\mathcal{C}(S)$. Then $\mathcal{T}(S)^+_{\text{sep}}$ is the preimage of $\mathcal{M}_g^c$ in $\mathcal{T}(S)^+$. The same reasoning as above shows that $\partial\mathcal{T}(S)^+_{\text{sep}}$ is homotopy equivalent to $\mathcal{C}(S)_{\text{sep}}$ and so $\partial\mathcal{T}(S)^+_{\text{sep}}$ is $(g-3)$-connected. It follows that we can construct a relative CW complex $(Z, \partial\mathcal{T}(S)^+_{\text{sep}})$ obtained from $\partial\mathcal{T}(S)^+_{\text{sep}}$ by attaching cells of dimension $\geq g-1$ in a $\Gamma(S)$-equivariant manner as to ensure that Z is contractible and no nontrivial element of $\Gamma(S)$ fixes a cell. Then Γ acts freely on Z (as it does on the

contractible space $\mathcal{T}(S)^+_{\mathrm{sep}}$) and so there is a Γ-equivariant homotopy equivalence $Z \to \mathcal{T}(S)^+_{\mathrm{sep}}$ relative to $\partial\mathcal{T}(S)^+_{\mathrm{sep}}$. It follows that we also have a homotopy equivalence $\Gamma\backslash Z \to \tilde{\mathcal{M}}^c_g$ relative to $\tilde{\Delta}^c_g$ and we conclude that $(\tilde{\mathcal{M}}^c_g, \tilde{\Delta}^c_g)$ is $(g-2)$-connected. If we take $\Gamma \subset \Gamma(S)$ (beyond being torsion free) normal and of finite index in $\Gamma(S)$, with finite quotient G, then $H_k(\mathcal{M}^c_g, \Delta^c_g; \mathbb{Q}) \cong H_k(\tilde{\mathcal{M}}^c_g, \tilde{\Delta}^c_g; \mathbb{Q})^G = 0$ for $k \leq g-2$. $\square$

A similar statement holds for the universal curve $\mathcal{M}_{g,1}$.

When $n > 1$, we need to come to terms with the fact that the separability notion has no good heriditary properties: if T is a closed surface, $A \subset T$ a compact 1-dimensional submanifold representing a simplex of $\mathcal{C}(T)$ and S a connected component of $T - A$, then a vertex of $\mathcal{C}(S)$ may split S, but not T. This happens precisely when the vertex in question separates two boundary components of ∂S that lie on the same connected component of $T - S$. So the basic object should be, what Andy Putman calls in [5], a partitioned surface: a closed surface minus a finite set, for which the removed set comes with a partition. This leads to the following definition.

Definition 1.4. Let N be the set of points of S at infinity (the cusps) and let P be a partition of N. We call a vertex of $\mathcal{C}(S)$ *separating relative to* P if a representative embedded circle $\alpha \subset S$ has the property that $S - \alpha$ has two connected components each of which meets N in a union of parts of P. We denote by $\mathcal{C}(S, P)$ the full subcomplex of $\mathcal{C}(S)$ spanned by such vertices.

So $\mathcal{C}(S, P) \subset \mathcal{C}_{\mathrm{sep}}(S)$ and we have equality when P is discrete or N is empty.

We shall prove Theorem 1.1 with induction simultaneously with

Theorem 1.5 ($A_{g,n}$)**.** *Suppose* $g > 0$ *and* $n = |N| > 1$. *Let* P *be a partition of* N. *Then* $\mathcal{C}(S, P)$ *is* $(g-2)$*-connected.*

Remark 1.6. I am indebted to Allen Hatcher for pointing out that the stronger version of Theorem 1.5 that I stated in a previous version was incorrect. Yet it may be that some such statement might hold. For instance, if $r(P)$ denotes the number of nonempty parts of P and $s(P)$ the number of parts with at least two elements, is it true that $\mathcal{C}(S, P)$ is $(g + r(P) + s(P) - 4)$-connected when $g > 0$ (as I claimed in the earlier version)? In case $g = 0$, $\mathcal{C}(S, P)$ is a complex of dimension $r(P) + s(P) - 4$. Is this $(r(P) + s(P) - 5)$-connected? In other words, is this complex spherical?

2. Proofs

Before we start off, we mention the following elementary fact that we will frequently use.

Lemma 2.1. *Let* X_i *be a* d_i*-connected space (*$d_i = -1$ *means* $X_i \neq \emptyset$*), where* $i = 1, \dots, k$. *Then the iterated join* $X_1 * \cdots * X_k$ *is* $(-2 + \sum_{i=1}^k (d_i + 2))$*-connected.*

Proof that ($A_{h,n}$) for $h < g$, all n, implies (A_g). So here $n \leq 1$. We must show that $\mathcal{C}_{\mathrm{sep}}(S)$ is $(g-3)$-connected. For $g < 2$, there is nothing to show and so we may assume that $g \geq 2$. A theorem of Harer [2, Thm. 1.2] asserts that $\mathcal{C}(S)$ is $(2g-3)$-connected. So it is certainly $(g-3)$-connected. Let $\mathcal{C}_k$ be the subcomplex of $\mathcal{C}(S)$ that is the union of $\mathcal{C}_{\mathrm{sep}}(S)$ and the k-skeleton of $\mathcal{C}(S)$. So $\mathcal{C}_{-1} = \mathcal{C}_{\mathrm{sep}}(S)$ and $\mathcal{C}_k = \mathcal{C}(S)$ for k large. Notice that a finite set of vertices of $\mathcal{C}(S)$ spans a simplex of $\mathcal{C}_k$ if and only if no more than $k+1$ of these are nonseparating. Hence a minimal simplex of $\mathcal{C}_k - \mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A \subset S$ with $k+1$ connected components, each of which is nonseparating. We prove that the link of such a simplex in $\mathcal{C}_k$ is a $(g-3)$-connected subcomplex of $\mathcal{C}_{k-1}$. This property implies that $|\mathcal{C}(S)|$ is obtained from $|\mathcal{C}_{\mathrm{sep}}(S)|$ by attaching cells of dimension $> g-3$ and since $\mathcal{C}(S)$ is $(g-3)$-connected, it then follows that $\mathcal{C}_{\mathrm{sep}}(S)$ is. Let $\{S_i\}_{i\in I}$ be the set of connected components of $S - A$. Notice that if g_i is the genus of S_i, then $g_i < g$. An Euler characteristic argument shows that

$$g - 1 \leq k + 1 + \sum_{i\in I}(g_i - 1).$$

We denote by N_i the set of connected components of A that bound S_i. The boundary components of the connected components of $S - S_i$ define a partition P_i of N_i. Since the connected components of A are nonseparating, $|N_i| \geq 2$. By our induction hypothesis $\mathcal{C}(S_i, P_i)$ is then $(g_i - 2)$-connected. The link of the k-simplex σ_A defined by A in $\mathcal{C}_k$ lies in $\mathcal{C}_{k-1}$ and can be identified with the $(|I|+1)$-fold join

$$\partial\sigma_A * \big(*_{i\in I}\mathcal{C}(S_i, \hat{P}_i)\big).$$

Since $\partial\sigma_A$ is a combinatorial $(k-1)$-sphere, it is $(k-2)$-connected and so by Lemma 2.1 the link in question has connectivity at least

$$(k-1) + \sum_{i\in I}(g_i - 1) + (|I| - 1) \geq g - 3 + (|I| - 1) \geq g - 3. \qquad \square$$

The proof of $A_{g,n}$ begins with a discussion. We now assume that $g > 0$ and $n \geq 2$ and denote by $\overline{S}$ the closed genus g surface obtained from S by adding its cusps.

Let $x \in N$. The goal is to compare $\mathcal{C}(S', P')$ with $\mathcal{C}(S, P)$. There is in general no forgetful map $\mathcal{C}(S, P) \to \mathcal{C}(S', P')$ because there will be vertices of $\mathcal{C}(S, P)$ that do not give vertices of $\mathcal{C}(S', P')$. Let us first identify this set of vertices.

Denote by $\Sigma_x \subset N - \{x\}$ is the set of $y \in N - \{x\}$ for which $\{x, y\}$ is a union of parts of P. In other words, if P_x denotes the part of P that contains x, then Σ_x is empty if P_x has more than 2 elements, equals $P_x - \{x\}$ if P_x is a 2-element set, and equals the set of $y \neq x$ for which P_y is a singleton, in case $P_x = \{x\}$. Then the set of vertices of $\mathcal{C}(S, P)$ that have no image in $\mathcal{C}(S', P')$ is precisely the set of vertices α of $\mathcal{C}_{\mathrm{sep}}(S)$ that for some $y \in \Sigma_x$ bound a disk neighborhood of $\{x, y\}$ in $S \cup \{x, y\}$ (so this set is empty if Σ_x is). Such a

disk neighborhood can be thought of as a regular neighborhood of an arc in $S \cup \{x, y\}$ connecting the two added cusps; this may help to explain why we have chosen to denote this set of vertices by $\mathrm{arc}_{(S,P)}(x)$. Denote by $\mathcal{C}(S, P)_x$ the full subcomplex of $\mathcal{C}(S, P)$ spanned by the vertices not in $\mathrm{arc}_{(S,P)}(x)$.

Observe that $\mathrm{arc}_{(S,P)}(x)$ is empty (so that $\mathcal{C}(S, P)_x = \mathcal{C}(S, P)$) is Σ_x is.

Lemma 2.2. *The link in $\mathcal{C}(S, P)$ of every vertex of $\mathrm{arc}_{(S,P)}(x)$ is a subcomplex of $\mathcal{C}(S, P)_x$ that projects isomorphically onto $\mathcal{C}(S', P')$.*

Proof. A vertex of $\mathrm{arc}_{(S,P)}(x)$ defines a $y \in \Sigma_x$ and (up to isotopy) a closed disk D in $S \cup \{x, y\}$ that is a neighborhood of $\{x, y\}$. We may identify the link in question with $\mathcal{C}(S \setminus D, P')$ and the latter clearly maps isomorphically onto $\mathcal{C}(S', P')$. □

Denote by $\tilde{P}$ the refinement of P which coincides with P on $N - P_x$ and partitions P_x further into $\{x\}$ and $P_x - \{x\}$. So $\tilde{P}' = P'$. It is clear that $\mathcal{C}(S, P)$ is a subcomplex of $\mathcal{C}(S, \tilde{P})$. Notice that $\mathrm{arc}_{(S,P)}(x) = \mathcal{C}(S, P) \cap \mathrm{arc}_{(S,\tilde{P})}(x)$ (we have $\mathrm{arc}_{(S,P)}(x) = \mathrm{arc}_{(S,\tilde{P})}(x)$ unless $|P_x| = 2$) and $\mathcal{C}(S, P)_x = \mathcal{C}(S, P) \cap \mathcal{C}(S, \tilde{P})_x$.

Lemma 2.3. *The simplicial map $f : \mathcal{C}(S, \tilde{P})_x \to \mathcal{C}(S', P')$ is a homotopy equivalence.*

Proof. Let us first observe the following. The map f is equivariant for the actions of mapping class group $\Gamma(S)$ (acting via $\Gamma(S')$ on $\mathcal{C}(S', P')$). The kernel of $\Gamma(S) \to \Gamma(S')$ may be identified with the fundamental group $\pi_1(S', x)$, so that the latter permutes the simplices of $\mathcal{C}(S, \tilde{P})_x$ that lie over any given simplex of $\mathcal{C}(S', P')$.

Let σ be a k-simplex of $\mathcal{C}(S', P')$. It is enough to prove that the preimage of $|\sigma|$ is contractible. Let σ be given by the compact one-dimensional submanifold A of S' with $k + 1$ connected components. This submanifold is unique up to isotopy in S'. If our representative A happens to avoid x, then it also defines a k-simplex of $\mathcal{C}(S, \tilde{P})_x$. If a component α of A passes through x, then we may define a $(k + 1)$-simplex in $\mathcal{C}(S, \tilde{P})_x$ by replacing α by the boundary of a thin regular neighborhood of α in S'. If we allow A to vary in its S'-isotopy class, then we thus produce all the simplices of $\mathcal{C}(S, \tilde{P})_x$ that map onto σ.

In order to describe the fiber of $|f|$ over the relative interior of $|\sigma|$, we choose a universal cover $\tilde{S}' \to S'$ of S'. Its total space is homeomorphic to an open disk. The preimage $\tilde{A}$ of a A in $\tilde{S}'$ is a closed one dimensional submanifold of which any connected component connects two boundary points of $\tilde{S}'$. (A clearer picture is perhaps obtained if we choose a hyperbolic structure on S' for which the points of N' are cusps and each component of A is a closed geodesic, for then $\tilde{S}'$ is a copy of the upper half plane and each component of $\tilde{A}$ is a complete geodesic in $\tilde{S}'$.) The fundamental group of a connected component of $S' - A$ maps injectively to the fundamental

group of S'. This implies that each connected component of $\tilde{S}' - \tilde{A}$ is simply connected and hence is contractible (in terms of hyperbolic geometry: it is a hyperbolic polygon with all its vertices—infinite in number—improper). So the closures of the connected components of $\tilde{S}' - \tilde{A}$ define a Leray covering of $\tilde{S}'$. If we denote its nerve by K_σ, then the geometric realization $|K_\sigma|$ has the homotopy type of $\tilde{S}'$, hence is contractible. In other words, K_σ is a tree. It is not hard to see that $|K_\sigma|$ can be identified with the fiber of $|f|$ over the relative interior of $|\sigma|$.

The situation over all of $|\sigma|$ is not much different: if $\tau \leq \sigma$ is obtained by omitting a number of connected components of A, then the associated partition of $\tilde{S}'$ becomes coarser and we have an obvious simplicial map $K_\sigma \to K_\tau$ of trees. The simplicial scheme over σ that we thus obtain is has still a contractible geometric realization. But this simplicial scheme is also the part of the barycentric subdivision of $\mathcal{C}(S, \tilde{P})$ that lies over σ. □

Corollary 2.4. *The complex $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ is canonically homotopy equivalent to the join $\mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S', P')$ (where $\mathrm{arc}_{(S,P)}(x)$ is discrete).*

Proof. The set of vertices of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ not in $\mathcal{C}(S,\tilde{P})_x$ is $\mathrm{arc}_{(S,P)}(x)$. The link of any such vertex is contained in $\mathcal{C}(S,\tilde{P})_x$ and by Lemma 2.2 that link projects isomorphically onto $\mathcal{C}(S', P')$. In view of Lemma 2.3 this implies that the inclusion of this link in $\mathcal{C}(S,\tilde{P})_x$ is also a homotopy equivalence. Hence the natural inclusion $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x \subset \mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S,\tilde{P})_x$ is a homotopy equivalence. The corollary follows. □

From now on we assume that A_g holds and that $A_{h,k}$ holds for all (h,k) smaller than (g,n) for the lexicographic ordering. Our goal is to prove $A_{g,n}$.

Lemma 2.5. *The pair $(\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x, \mathcal{C}(S,P))$ is $(g-1)$-connected.*

Proof. If $P_x = \{x\}$, then $\tilde{P} = P$ and there is nothing to show. We therefore assume that P_x has more than one element. Denote by $\mathcal{C}_k$ the subcomplex of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ that is the union of $\mathcal{C}(S,P)$ and the k-skeleton of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$. So $\mathcal{C}_{-1} = \mathcal{C}(S,P)$ and $\mathcal{C}_k = \mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ for k large. A finite set of vertices of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ spans a simplex of $\mathcal{C}_k$ if and only if no more than $k+1$ of these separate x from $P_x - \{x\}$. So a minimal simplex of $\mathcal{C}_k - \mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A \subset S$ with $k+1$ connected components, each of which separates x from $P_x - \{x\}$ (the graph that is associated to A is then a string with $k+2$ nodes). We prove that the link of such a simplex in $\mathcal{C}_{k-1}$ is $(g-2)$-connected if $g > 0$. This will suffice.

We enumerate the connected components of A as $\alpha_0, \dots, \alpha_k$ and the connected components of $S - A$ as $S_0, \dots, S_{k+1}$ such that α_i is a boundary component of S_i and S_{i+1} and so that S_0 resp. S_{k+1} is punctured by x resp. $P_x - \{x\}$. So if we put $\hat{N} := N \sqcup \{0, \dots, k\}$, then the set of cusps of S_i is naturally indexed by a subset $\hat{N}_i$ of $\hat{N}$ and these subsets partition $\hat{N}$. Observe

that $|\hat{N}_i| \geq 2$ for every i. Denote by P_i the partition of $(N - P_x) \cap S_i$ that is simply the restriction of P and denote by $\hat{P}_i$ the partition of $\hat{N}_i$ that is on $(N - P_x) \cap S_i$ equal to P_i and has the remainder (i.e., $(\{1, \dots, k\} \cup P_x)|S_i$) as a single part. So this new part is $\{x\} \cup \{0\}$ for $i = 0$, $\{i-1, i\}$ for $0 < i < k+1$ and $P_x - \{x\}) \cup \{k\}$ for $i = k+1$.

The reason for introducing these partitions is that we can now observe that the link of the k-simplex σ_A defined by A in $\mathcal{C}_k$ lies in $\mathcal{C}_{k-1}$ and can be identified with the iterated join

$$\partial\sigma_A * \mathcal{C}(S_0, \hat{P}_0) * \cdots * \mathcal{C}(S_{k+1}, \hat{P}_{k+1}).$$

It is then enough to show that this join is $(g-2)$-connected for $g > 0$. Since $\partial\sigma_A$ is a $(k-1)$-sphere, it is $(k-2)$-connected. The connectivity of a factor $\mathcal{C}(S_i, \hat{P}_i)$ with $g_i > 0$ is at least $g_i - 2$. So by Lemma 2.1 the connectivity of the above join is at least $-2 + k + \sum_{\{i:g_i>0\}} g_i = g + k - 2 \geq g - 2$. □

Proof of $(A_{g,n})$. We must show that $\mathcal{C}(S, P)$ is $(g-2)$-connected. In view of Lemma 2.5 is suffices to show that $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ has that property.

If $\mathrm{arc}_{(S,P)}(x) = \emptyset$, then $n > 2$ and so our induction hypothesis implies that $\mathcal{C}(S', P')$ is $(g-2)$-connected by $A_{g,n-1}$. It follows from Corollary 2.4 that $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is homotopy equivalent to $\mathcal{C}(S', P')$ and hence is $(g-2)$-connected.

If $\mathrm{arc}_{(S,P)}(x) \neq \emptyset$, then we may have $n = 2$. At least we know that $\mathcal{C}(S', P')$ is $(g-3)$-connected (invoke A_g if $n = 2$). But since $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is homotopy equivalent to $\mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S', P')$ (by Corollary 2.4), it is $(g-2)$-connected. □

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